A simplified explanation of how to get unstable spirals

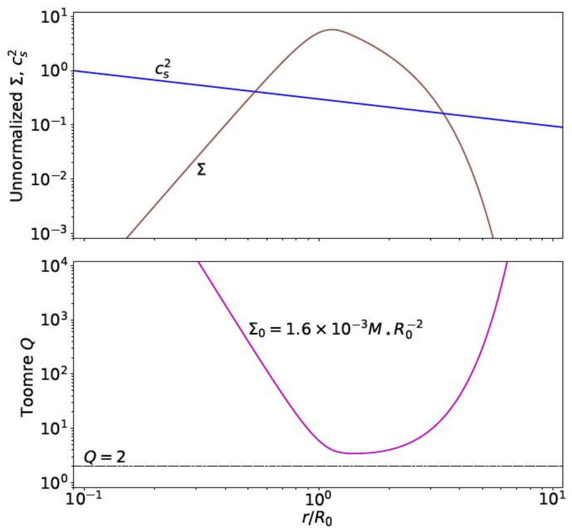
The full derivation can be found in Section 3 and Appendix A-C of our paper.

Related theoretical discussion can also be found in Lin (2015; arXiv:1502.02662) and Lee et al (2019; arXiv:1811.11758)

Big picture:

Disk with strong self-gravity

Fast disk cooling



traps spiral eccentric modes

drives spiral eccentric modes to grow

β-cooling with small β
OR
Locally isothermal disk

Instability of spiral eccentric mode

Disk with strong self-gravity traps spiral eccentric modes

Basic equation of the disk eccentric mode:

$$2r^3\Omega_{\rm K}\Sigma\omega E = [\frac{i\beta}{i\beta+1}M_{\rm adi} + \frac{1}{i\beta+1}M_{\rm iso} + M_{\rm dsg} + M_\beta]E.$$

The four operators are :

$$M_{\rm adi}E = \frac{d}{dr}\left(\gamma r^3 P\frac{dE}{dr}\right) + r^2\frac{dP}{dr}E,$$

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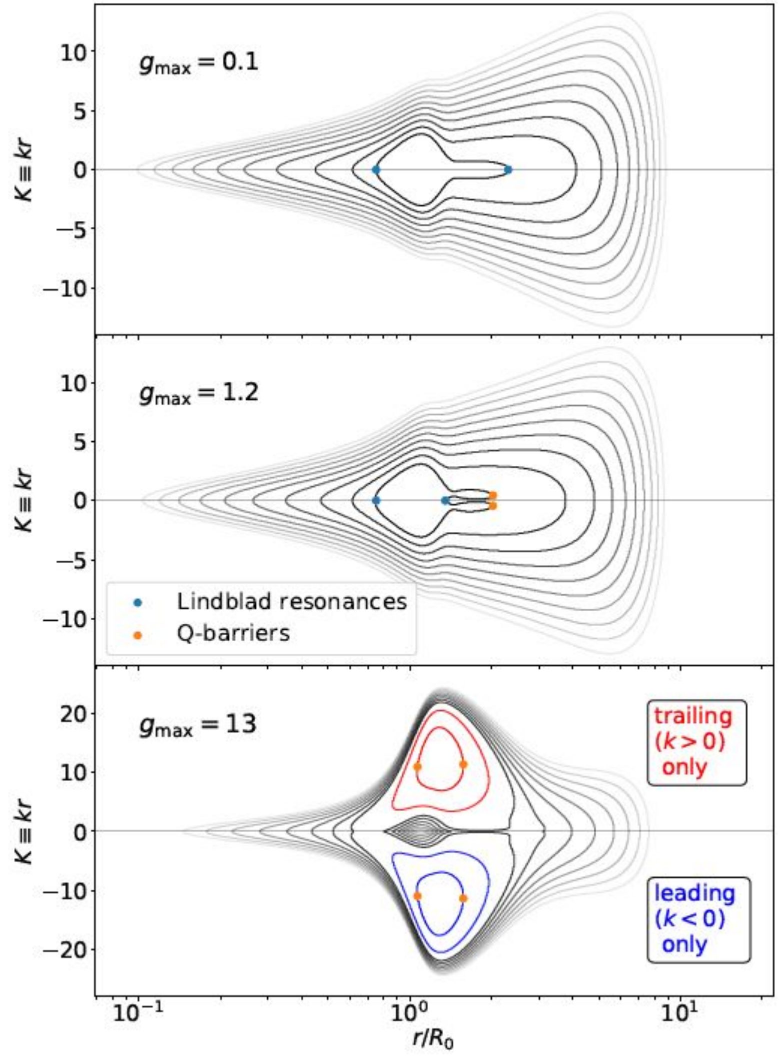

Disk with strong self-gravity traps spiral eccentric modes

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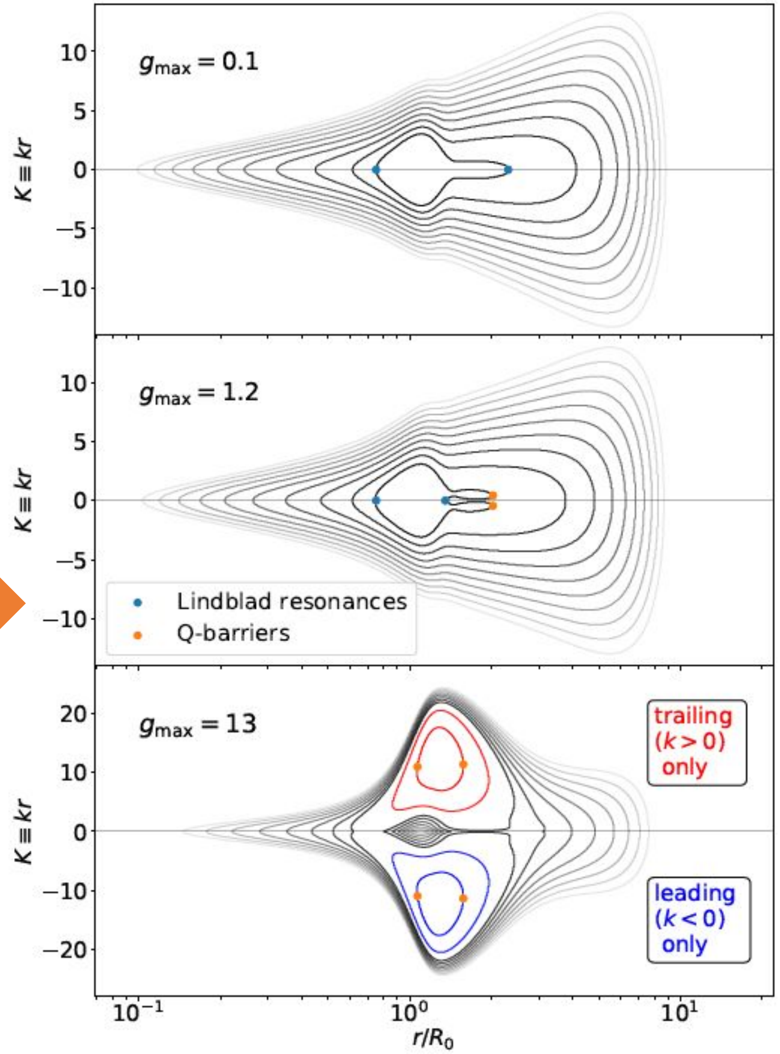
Dispersion relation map (DRM) for eccentric modes:

- Derived from the disk eccentric mode equation.
- $g(r) = \frac{\pi G \Sigma r}{c_s^2}$: larger g means stronger self-gravity against pressure.
- X-axis: r coordinate of the disk;
  Y-axis: wave number k.
- Closed curves: wave frequency $\omega(k, r)$ contours that represents standing eccentric waves (which satisfy the quantum condition).
- Red/Blue contours: trailing/leading spiral modes,
  which **only appears when self-gravity is strong.**



Fast disk cooling drives the growth of spiral eccentric modes

- Consider the disk as a collection of many eccentric rings.

- For a ring of radius r, its eccentricity implies an angular momentum deficit (AMD):

$$\ell(r,e) = \sqrt{GM_* r(1-e^2)}$$

$$\rightarrow \quad AMD(r,E) = \frac{1}{2} L_K |E|^2 2\pi r \quad \text{for small } |E| \text{ with } L_K = \Sigma\Omega_K r^2.$$

- To change the $|E|$ of an eccentric mode is to change to the total AMD of all rings that form a disk:

$$\int \pi r L_K |E|^2 dr$$

Fast disk cooling drives the growth of spiral eccentric modes

We can derive the time evolution of the disk AMD using our basis equation of eccentric mode:

$$\frac{d}{dt}\int 2rL_K|E|^2 dr = \int 2r^3\Omega_K \Sigma E^* \frac{\partial E}{\partial t} dr + c.c.$$

where $c.c.$ stands for the complex conjugate and $\frac{\partial E}{\partial t} = i\omega E$ for eccentric modes is determined by

$$2r^3\Omega_{\rm K}\Sigma\omega E = [\frac{i\beta}{i\beta+1}M_{\rm adi} + \frac{1}{i\beta+1}M_{\rm iso} + M_{\rm dsg} + M_\beta]E.$$

Fast disk cooling drives the growth of spiral eccentric modes

Which term in the mode equation can change the angular momentum (i.e., mode amplitude $|E|$)?

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The last matrix is always β unless beta is not a constant. Let us make our lives easier by assume it is zero here.

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Gravity from the axisymmetric background:
Does not change the local AMD

Mutual gravity between the eccentric rings:
Does not change the total AMD

The self-gravity term does not cause mode instability

Fast disk cooling drives the growth of spiral eccentric modes

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Axisymmetric pressure on the eccentric rings: Causes the precession of the rings, but does not change the AMD

Dispersive eccentricity propagation: Does not change the total AMD inside our wave-trapping boundaries

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Fast disk cooling drives the growth of spiral eccentric modes

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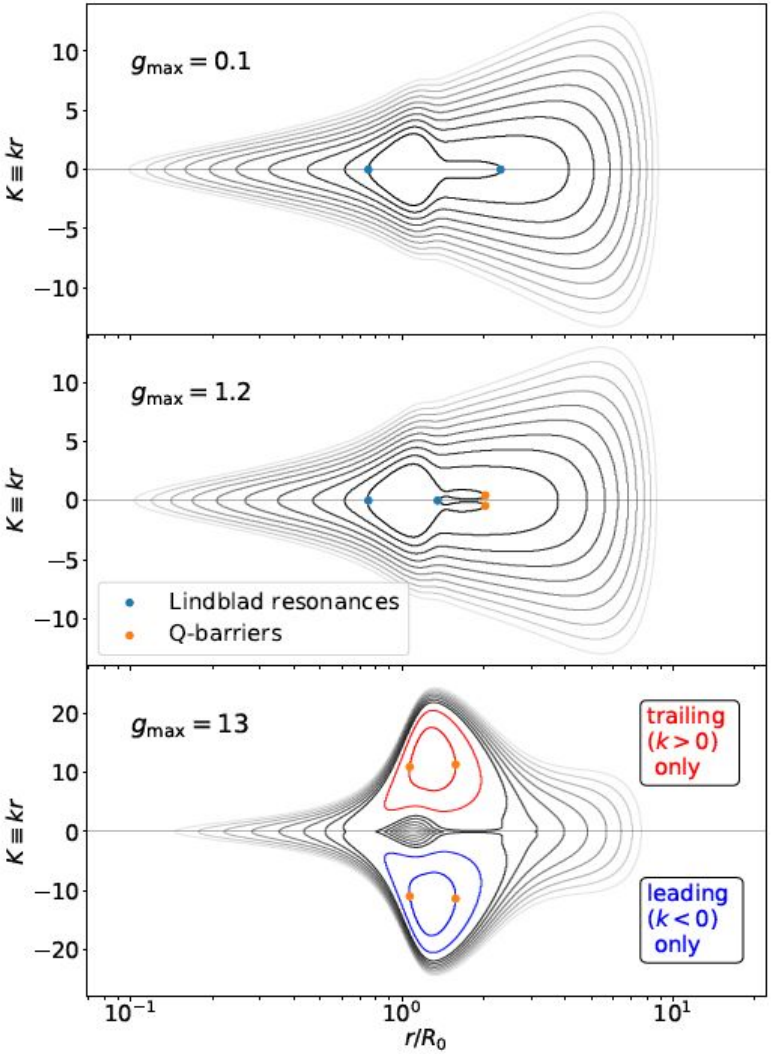
As fluid material is perturbed, the background disk imposes its locally temperature to the perturbation. It can possibly lead to angular momentum exchange between the mode and the background.

This angular momentum exchange is also derived in Lin & Papaloizou (2011, arXiv:1103.5036)

Fast disk cooling drives the growth of spiral eccentric modes



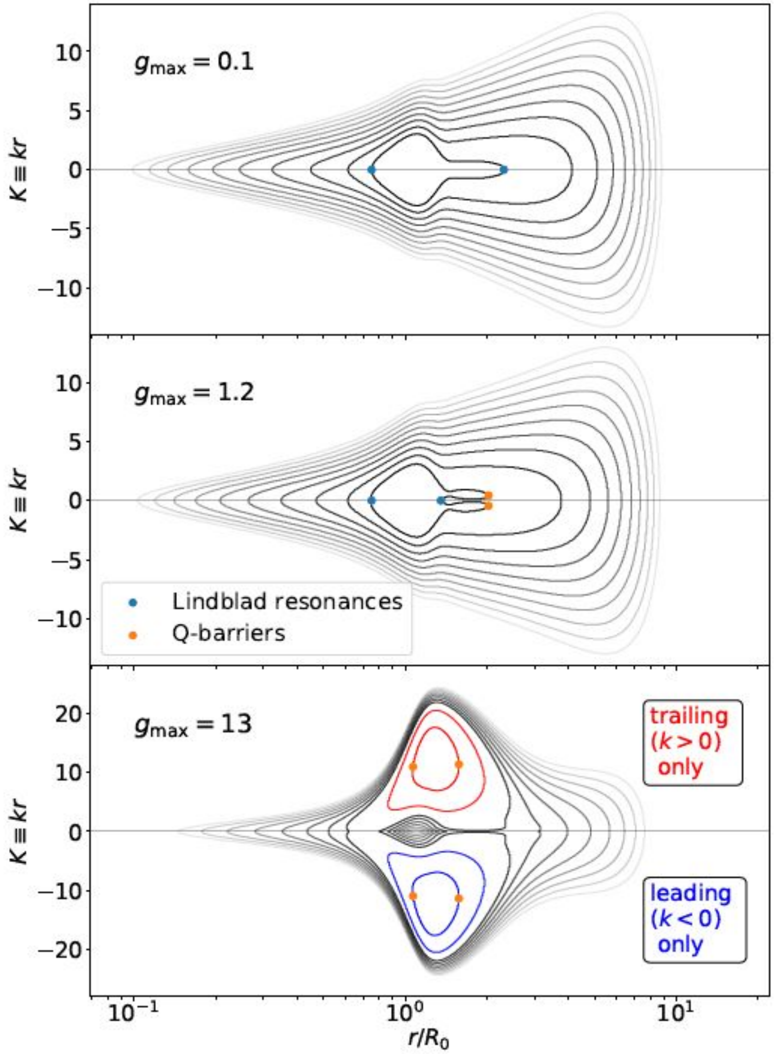
Assess the effect of the temperature gradient term to AMD:

$$\frac{d}{dt}\int 2rL_K|E|^2 dr \Leftarrow i\int E^* \frac{\partial}{\partial r}\left(\Sigma \frac{dc_s^2}{dr} r^3 E\right) dr + c.c.$$

For the elliptical modes (top and middle panels), $E \propto e^{ikr} + e^{-ikr}$, so the in term in the bracket is real. Remember $c.c.$ is the complex conjugate, so the RHS is zero.

For the spiral mode, $\boldsymbol{E \propto e^{ikr}}$ where k is always positive or negative, so the equation is non-zero. That means the mode is always gaining or losing angular momentum, causes a growth or damping of the mode depending on the sign of k.

Fast disk cooling drives the growth of spiral eccentric modes



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For the spiral mode, $E \propto e^{ikr}$ where k is always positive or negative. One can integrate the RHS by parts and finish estimate the growth/damping rate as:

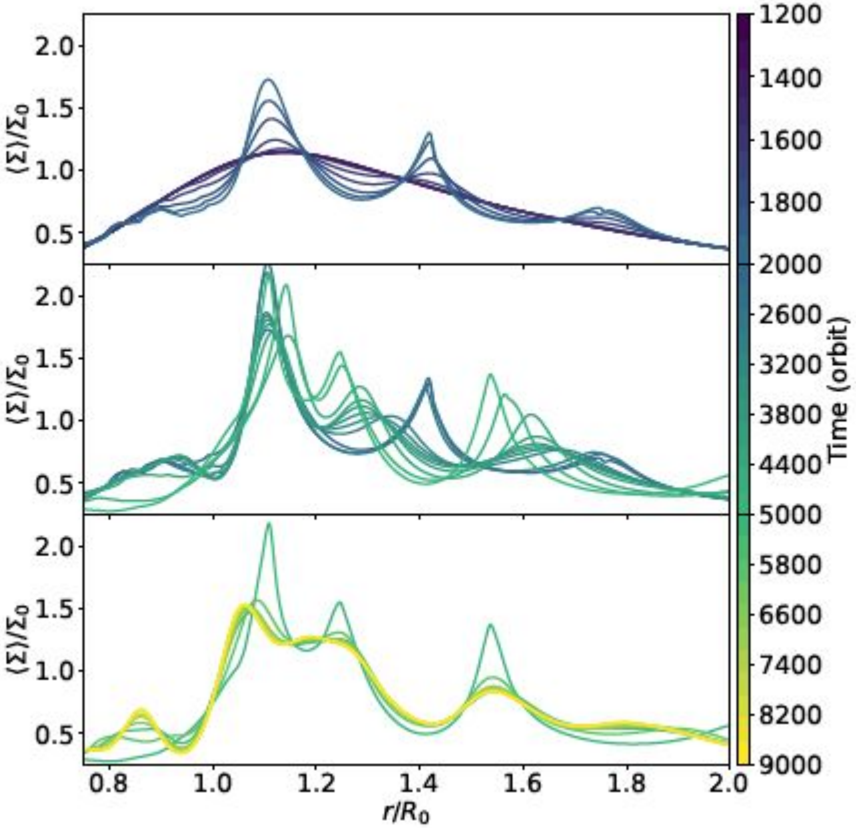
$$\frac{d}{dt}\int 2rL_K|E|^2 dr = \int 2rL_K \Omega h^2 g \frac{\mathrm{dln}c_s^2}{\mathrm{dln}r} |E|^2 dr,$$

so roughly speaking, the growth rate is:

$$\frac{d|E|}{dt} = \frac{gh^2\Omega}{2},$$

which is, to the zeroth-order the same as what we found in the paper.

Final words:



Due to the angular momentum exchange with the mode, the (axisymmetric) background also need to rearrange its density profile. That is why rings can appear and how the instability saturates.

See Section 3.2 and Appendix D of our paper for details.